EINSTEIN'S THEORY OF RELATIVITY CONSID-
ERED FROM THE EPISTEMOLOGICAL
STANDPOINT*

VI. EUCLIDEAN AND NON-EUCLIDEAN GEOMETRY

I N THE preceding considerations, however, we have
taken up only incidentally an achievement of the general
theory of relativity, which, like scarcely a second, seems
to involve a "revolution of thought." In the working out
of the theory, it is seen that the previous Euclidean meas-
urements are not sufficient; the development of the theory
can only take place by our going from the Euclidean con-
tinuum, which was still taken as a basis by the special the-
ory of relativity, to a non-Euclidean four-dimensional
space-time continuum and seeking to express all relations
of phenomena in it. Thus a question seems answered
physically which had concerned the epistemology of the
last decades most vitally and which had been answered
most diversely within it. Physics now proves not only the
possibility, but the reality of non-Euclidean geometry; it
shows that we can only understand and represent theoreti-
cally the relations, which hold in "real" space, by repro-
ducing them in the language of a four-dimensional non-
Euclidean manifold.

The solution of this problem from the side of physics
was, on the one hand, for a long time hoped for as keenly,

* Translated by W. C. and M. C. Swabey.

as, on the other hand, its possibility was vigorously denied. Even the first founders and representatives of the doctrine of non-Euclidean geometry sought to adduce experiment and concrete measurement in confirmation of their view. If we can establish, they inferred, by exact terrestrial or astronomical measurements, that in triangles with sides of very great length the sum of the angles differs from two right angles, then empirical proof would be gained that in "our" empirical space the propositions not of Euclidean geometry, but of one of the others were valid. Thus, *e. g.*, Lobatschefsky, as is known, used a triangle E_1 E_2 S, whose base E_1 E_2 was formed by the diameter of the orbit of the earth and whose apex S was formed by Sirius and believed that he could, in this way, prove empirically a possible constant curvature of our space. (48.) The fallacy in method of any such attempt must be obvious, however, to any sharper epistemological analysis of the problem and it has been pointed out from the side of the mathematicians with special emphasis by H. Poincaré. No measurement, as Poincaré objects with justice, is concerned with space itself but always only with the empirically given and physical objects in space. No experiment therefore can teach us anything about the *ideal* structures, about the straight line and the circle, that pure geometry takes as a basis; what it gives us is always only knowledge of the relations of material things and processes. The propositions of geometry are therefore neither to be confirmed nor refuted by experience. No experiment will ever come into conflict with the postulates of Euclid; but, on the other hand, no experiment will ever contradict the postulates of Lobatschefsky. For granted, that some experiment could show us a variation in the sums of the angles of certain very great triangles, then the conceptual representation of this fact would never need to consist in, and methodologi-

cally could not consist in, changing the axioms of geom-
etry, but rather in changing certain hypotheses concern-
ing physical things. What we would have *experienced,*
in fact, would not be another structure of space, but a new
law of optics, which would teach us that the propagation
of light does not take place in strictly rectilinear fashion.
"However, we turn and twist," Poincaré therefore con-
cludes, "it is impossible to attach a rational meaning to
empiricism in geometry." (72, p. 92ff.) If this decision
holds and if it can be proved, on the other hand, that among
all possible self-consistent geometries the Euclidean pos-
sesses a certain advantage of "simplicity" since it defines
the minimum of those conditions under which experience
is possible in general, there would then be established for
it an exceptional position from the standpoint of the
critique of knowledge. It would be seen that the different
geometries, which are equivalent to each other from a
purely formal standpoint, as regards their logical conceiv-
ability, are yet distinguished in their fruitfulness in the
founding of empirical science. "The geometries are dis-
tinguished from each other in principle," one can conclude,
"only by reference to their epistemological relation to the
concept of experience; for this relation is positive only in
the case of the Euclidean geometry."[29]

In connection, however, with the new development of
physics in the general theory of relativity, this epistemo-
logical answer seems to become definitively untenable. Again
and again the fact has been appealed to in the controversy
concerning the epistemological justification of the differ-
ent geometries that what determines value must not be
sought in formal but in transcendental logic; that the com-
patibility of a geometry with experience is not involved but
rather its "positive fruitfulness," *i. e.,* the " founding of

[29] *Cf.* Hönigswald (32) ; on the following *cf.* Bauch (1), p. 126ff.

experience," that it can give. And this latter was thought to be found in Euclidean geometry. The latter appeared as the real and unique "foundation of possibility of knowledge of reality," the others, on the contrary, always as only the foundations of the possible. But with regard to the extraordinary rôle that the concepts and propositions of Riemannian geometry played in the grounding and construction of Einstein's theory of gravitation, this judgment cannot be supported. Supported by the same logical criterion of value, one now seems forced rather to the opposite conclusion: non-Euclidean space is alone "real," while Euclidean space represents a mere abstract possibility. In any event, the logic of the exact sciences now finds itself placed before a new problem. The fact of the fruitfulness of non-Euclidean geometry for physics can no longer be contested, since it has been verified, not only in particular applications, but in the structure of a complete new system of physics; what is in question is the explanation to be given to this fact. And here we are first forced to a negative decision, which is demanded by the first principles of the theory of relativity. Whatever meaning we may ascribe to the idea of non-Euclidean geometry for physics, for purely empirical thought, the assertion has lost all meaning for us that any space, whether Euclidean or non-Euclidean, is the "real" space. Precisely this was the result of the general principle of relativity, that by it "the last remainder of physical objectivity" was to be taken from space. Only the various relations of measurement within the physical manifold, within that inseparable correlation of space, time, and the physically real object, which the theory of relativity takes as ultimate, are pointed out; and it is affirmed that these relations of measurement find their simplest exact mathematical expression in the language of non-Euclidean geometry. This language, how-

ever, is and remains purely ideal and symbolic, precisely as, rightly understood, the language of Euclidean geometry could alone be. The reality which alone it can express is not that of things, but that of laws and relations. And now we can ask, epistemologically, only one question: whether there can be established an exact relation and co-ördination between the symbols of non-Euclidean geometry and the empirical manifold of spatio-temporal "events." If physics answers this question affirmatively, then epistemology has no ground for answering it negatively. For the "a priori" of space that it affirms as the condition of every physical theory involves, as has been seen, no assertion concerning any definite particular structure of space in itself, but is concerned only with that function of "spatiality" in general, that is expressed even in the general concept of the linear element ds as such, quite without regard to its character in detail.

If it is seen thus, that the determination of this element as is done in Euclidean geometry, does not suffice for the mastery of certain problems of knowledge of nature then nothing can prevent us, from a methodological standpoint, from replacing it by another measure, in so far as the latter proves to be necessary and fruitful physically. But in either case one must guard against taking the "preestablished harmony between pure mathematics and physics," that is revealed to us in increasing fulness and depth in the progress of scientific knowledge, as a naïve copy theory. The structures of geometry, whether Euclidean or non-Euclidean, possess no immediate correlate in the world of *existence*. They exist as little physically in things as they do psychically in our "presentations" but all their "being," *i. e.,* their validity and truth, consists in their ideal *meaning*. The existence, that belongs to them by virtue of their definition, by virtue of a pure logical act of

assumption is, in principle, not to be interchanged with any sort of empirical "reality." Thus also the applicability, which we grant to any propositions of pure geometry, can never rest on any direct coinciding between the elements of the ideal geometrical manifold and those of the empirical manifold. In place of such a sensuous congruence we must substitute a more complex and more thoroughly mediate relational system. There can be no copy or correlate in the world of sensation and presentation for what the points, the straight lines and the planes of pure geometry signify. Indeed, we cannot in strictness speak of any degree of similarity, of greater or less difference of the "empirical" from the ideal, for the two belong to fundamentally different species. The theoretical relation, which science nevertheless establishes between the two, consists merely in the fact, that it, while granting and holding fast to the difference in content of the two series, seeks to establish a more exact and perfect correlation between them. All verification, which the propositions of geometry can find in physics, is possible only in this way. The particular geometrical truths or particular axioms, such as the principle of parallels, can never be compared with particular experiences, but we can always only compare with the whole of physical experience the whole of a definite system of axioms. What Kant says of the concepts of the understanding in general, that they only serve "to make letters out of phenomena so that we can read them as experiences" holds in particular of the concepts of space. They are only the letters, which we must make into words and propositions, if we would use them as expressions of the laws of experience. If the goal of harmony is not reached in this indirect way, if it appears that the physical laws to which observation and measurement lead us cannot be represented and expressed with sufficient exactitude and simplicity by a given system

of axioms, then we are free to determine which of the
two factors we shall subject to a transformation to reëstab-
lish the lost harmony between them.   Before thought ad-
vances to a change of one of its "simple" geometrical laws
it will first make the complex physical conditions that enter
into the measurement responsible for the lack of agree-
ment; it will change the "physical" factors before the "geo
metrical."   If this does not lead to the goal and if it is seen,
on the other hand, that surprising unity and systematic
completeness can be reached in the formulation of the "laws
of nature" by acceptng an altered conception of geometrical
methods, then in principle there is nothing to prevent such
a change.   For if we conceive the geometrical axioms, not
as copies of a given reality, but as purely ideal and con-
structive structures, then they are subjected to no other
law than is given them by the *system* of thought and knowl-
edge.   If the latter proves to be realizable in a purer and
more perfect form by our advancing from a relatively sim-
pler geometrical system to a relatively more complex, then
the criticism of knowledge can raise no objection from its
standpoint.   It will be obliged to affirm only this: that here
too "no intelligible meaning can be gained" for empiricism
in geometry.   For here, too, experience does not *ground*
the geometrical axioms, but it only makes from among
them, as various logically possible systems, of which each
one is derived strictly rationally, a certain selection as to
their concrete use, as to the interpretation of phenomena.[30]
Here, too, Platonically speaking, phenomena are measured
by Ideas, by the foundations of geometry, and these latter
are not directly read out of the sensuous phenomena.

But when one grants to non-Euclidean geometry in this
sense meaning and fruitfulness for physical experience, the
general methodic difference can and must be urged, that

[30] On this relation of the problem of metageometry to the problem of
"experience," *cf.* esp. Albert Görland (28, p. 324ff.)

still remains between it and Euclidean geometry. This difference can no longer be taken from their relation to experience, but it must be recognized as based on certain "inner" moments, *i. e.*, on general considerations of the *theory of relations*. A special and exceptional logical position, a fundamental simplicity of ideal structure, can be recognized in Euclidean geometry even if it must abandon its previous sovereignty within physics. And here it is precisely the fundamental doctrine of the general theory of relativity, that, translated back from the language of logic and general methodology, can establish and render intelligible this special position. Euclidean geometry rests on a definite axiom of relativity, which is peculiar to it. As the geometry of space of a constant curvature O, it is characterized by the thorough-going relativity of all places and magnitudes. Its formal determinations are in principle independent of any absolute determinations of magnitude. While, *e. g.*, in the geometry of Lobatschefsky, the sum of the angles of a rectilinear triangle is different from 180° and indeed the more so, the more the surface area of the triangle increases, the absolute magnitude of the lines enters into none of the propositions of Euclidean geometry. Here for every given figure a "similar" can be constructed; the particular structures are grasped in their pure "quality," without any definite "quantum," any absolute value of number and magnitude, coming into consideration in their definition. This indifference of Euclidean structures to all absolute determinations of magnitude and the freedom resulting here of the particular points in Euclidean space of all determinations and properties, form a logically positive characteristic of the latter. For the proposition, *omnis determinatio est negatio,* holds here too. The assumption of the indeterminate serves as the foundation for the more complex assumptions and deter-

minations, that can join on to it. In this sense, Euclidean geometry is and remains the "simplest," not in any practical, but in a strictly logical meaning; Euclidean space is, as Poincaré expresses it, "simpler not merely in consequence of our mental habits or in consequence of any direct intuition, which we possess of it, but it is in itself simpler, just as a polynomial of the first degree is simpler than a polynomial of the second degree." (72, p. 67.) This logical simplicity belonging to Euclidean space in the system of our intellectual meanings wholly independently of its relations to experience, is shown, *e. g.,* in the fact that we can make any "given" space, that possesses any definite curvature, into Euclidean by regarding sufficiently small fields of it from which the difference conditioned by the curvature disappears. Euclidean geometry shows itself herein as the real geometry of infinitely small areas, and thus as the expression of certain elementary relations, which we take as a basis in thought, although we advance from them in certain cases to more complex forms.

The development of the general theory of relativity leaves this methodic advantage of Euclidean geometry unaffected. For Euclidean measurements do not indeed hold in it absolutely but they hold for certain "elementary" areas, which are distinguished by a certain simplicity of physical conditions. The Euclidean expression of the linear element shows itself to be unsatisfactory for the working out of the fundamental thought of the general theory of relativity, since it does not fulfill the fundamental demand of retaining its form in every arbitrary alteration of the system of reference. It must be replaced by the *general* linear element ($ds^2 = \sum_1^4 g_{\mu\nu} dx_\mu dx_\nu$), which satisfies this demand. If, however, we consider infinitely small four-dimensional fields, it is expressly demanded that the pre-

suppositions of the special theory of relativity, and thus its Euclidean measurements shall remain adequate for them. The form of the universal linear element here passes over into the Euclidean element of the special theory when the ten magnitudes g, which occur in this as functions of the coördinates of particular points assume definite constant values. The physical explanation of this relation, however, consists in that the magnitudes $g_{\mu\nu}$ are recognized as those which describe the gravitational field with reference to the chosen system of reference. The condition, under which we can pass from the presuppositions of the general theory of relativity to the special theory, can accordingly be expressed in the form that we only consider regions within which abstraction can be made from the effects of fields of gravitation. This is always possible for an infinitely small field and it holds further for finite fields in which, with appropriate choice of the system of reference, the body considered undergoes no noticeable acceleration. As we see, the variability of the magnitudes $g_{\mu\nu}$, which expresses the variation from the homogeneous Euclidean form of space, is recognized as based on a definite *physical* circumstance. If we consider fields in which this circumstance is absent or if we cancel it in thought, we again stand within the Euclidean world. Thus the assertion of Poincaré that all physical theory and physical measurement can prove absolutely nothing about the Euclidean or non-Euclidean character of *space,* since it is never concerned with the latter but only with the properties of *physical reality in space* remains thus entirely in force. The abstraction (or, better expressed, the pure function) of homogeneous Euclidean space is not destroyed by the theory of relativity, but is only known as such through it more sharply than before.

In fact, the pure meaning of geometrical concepts is not limited by what this theory teaches us about the conditions of measurement. These concepts are indeed, as is seen now anew, neither an empirical datum nor an empirical *dabile,* but their ideal certainty and meaning is not in the least affected thereby. It is shown that in fields where we have to reckon with gravitational effects of a definite magnitude, the preconditions of the ordinary methods of measurement fall aside, that here we can no longer use "rigid bodies" as measures of length, nor ordinary "clocks" as measures of time. But this change of relations of measurement does not affect the calculation of space, but the calculation of the physical relation between the measuring rods and rays of light determined by the field of gravitation. (*Cf.* 83, p. 85ff.) The truths of Euclidean geometry would only be also affected if one supposed that these propositions themselves are nothing but generalizations of empirical observation, which we have established in connection wth fixed bodies. Such a supposition, however, epistemologically regarded, would amount to a *petitio principii.* Even Helmholtz, who greatly emphasizes the empirical origin of the geometrical axioms occasionally refers to another view, which might save their purely ideal and "transcendental" character. The Euclidean concept of the straight line might be conceived not as a generalization from certain physical observations, but as a purely ideal concept, to be confirmed or refuted by no experience, since we would have to decide by it whether any bodies of nature were to be regarded as fixed bodies. But, as he objects, the geometrical axioms would then cease to be synthetical propositions in Kant's sense, as they would only affirm something that would follow analytically from the concepts of the fixed geometrical structures necessary to measurement. (30a, II, 30.) It is, however, overlooked by

this objection that there are *fundamentally synthetic forms of unity* besides the form of analytic identity, which Helmholtz has here in mind and which he contrasts with the empirical concept as if the form of analytic identity were unique, and that the axioms of geometry belong precisely to the former.   Assumptions of this sort refer to the object in so far as in their totality they "constitute" the object and render possible knowledge of it; but none of them, taken for itself, can be understood as an assertion concerning things or relations of things.   Whether they fulfill their task as moments of empirical knowledge can be decided always only in the indicated indirect way: by using them as building-stones in a theoretical and constructive system, and then comparing the consequences, which follow from the latter, with the results of observation and measurement.   That the elements, to which we must ascribe, methodologically, a certain "simplicity," must be adequate for the interpretation of the laws of nature, can not be demanded *a priori*.   But even so, thought does not simply give itself over passively to the mere *material* of experience, but it develops out of itself new and more complex *forms* to satisfy the demands of the empirical manifold.

If we retain this general view, then one of the strangest and, at first appearance, most objectionable results of the general theory of relativity receives a new light.   It is a necessary consequence of this theory that in it one can no longer speak of an immutably given geometry of measurement, which holds once for all for the whole world. Since the relations of measurement of space are determined by the gravitational potential and since this is to be regarded as in general changeable from place to place, we cannot avoid the conclusion that there is in general no unitary "geometry" for the totality of space and reality, but that, according to the specific properties of the field of

gravitation at different places, there must be found different forms of geometrical structure. This seems, in fact, the greatest conceivable departure from the idealistic and Platonic conception of geometry, according to which it is the "science of the eternally existent," knowledge of what always "is in the same state" (ἀεὶ κατὰ ταὐτὰ ὡσαύτως ἔχον). Relativism seems here to pass over directly into the field of logic; the relativity of places involves that of geometrical truth. And yet this view is, on the other hand, only the sharpest expression of the fact that the problem of space has lost all ontological meaning in the theory of relativity. The purely methodological question has been substituted for the question of being. We are no longer concerned with what space "is" and with whether any definite character, whether Euclidean, Lobatschefskian or Riemannian, is to be ascribed to it, but rather with what use is to be made of the different systems of geometrical presuppositions in the interpretation of the phenomena of nature and their dependencies according to law. If we call any such system a particular "space," then indeed we can no longer attempt to grasp all of these spaces as intuitive parts to be united into an intuitive whole. But this impossibility rests fundamentally on the fact that we have here to do with a problem, which as such stands outside the limits of intuitive representation in general. The space of pure intuition is always only *ideal,* being only the space constructed according to the laws of this intuition, while here we are not concerned with such ideal syntheses and their unity, but with the relations of measurement of the empirical and the physical. These relations of measurement can only be gained on the basis of natural laws, *i. e.,* by proceeding from the dynamic dependency of phenomena upon each other, and by permitting phenomena to determine their positions reciprocally in the space-time mani-

fold by virtue of this dependency. Kant too decisively urged that this form of dynamic determination did not belong to intuition as such, but that it is the "rules of the understanding" which alone give the existence of phenomena synthetic unity and enable them to be collected into a definite concept of experience. (*Cf.* above, p. 79.) The step beyond him, that we have now to make on the basis of the results of the general theory of relativity, consists in the insight that geometrical axioms and laws of other than Euclidean form can enter into this determination of the understanding, in which the empirical and physical world arises for us, and that the admission of such axioms not only does not destroy the unity of the world, *i. e.,* the unity of our experiential concept of a total order of phenomena, but first truly grounds it from a new angle, since in this way the particular laws of nature, with which we have to calculate in space-time determination, are ultimately brought to the unity of a supreme principle,—that of the universal postulate of relativity. The renunciation of intuitive simplicity in the picture of the world thus contains the guarantee of its greater intellectual and systematic completeness. This advance, however, can not surprise us from the epistemological point of view; for it expresses only a general law of scientific and in particular of physical thought. Instead of speaking ontologically of the being or indeed of the coexistence of diversely constituted "spaces," which results in a tangible contradiction, the theory of relativity speaks purely methodologically of the possibility of necessity of applying different measurements, *i. e.,* different geometrical conceptual languages in the interpretation of certain physical manifolds. This possible application tells us nothing concerning the "existence" of spaces, but merely indicates that by an appropriate choice of geometrical presuppositions certain physical relations,

such as the field of gravitation or the electromagnetic field, can be described.

The connection between the purely conceptual thought, involved in the working out of the general doctrine of the manifold and order, and physical empiricism (*Empirie*) here receives a surprising confirmation. A doctrine, which originally grew up merely in the immanent progress of pure mathematical speculation, in the ideal transformation of the hypotheses that lie at the basis of geometry, now serves directly as the form into which the laws of nature are poured. The same functions, that were previously established as expressing the metrical properties of non-Euclidean space, give the equations of the field of gravitation. These equations thus do not need for their establishment the introduction of new unknown forces acting at a distance, but are derived from the determination and specialization of the general presuppositions of measurement. Instead of a new complex of things, the theory is satisfied here by the consideration of a new general complex of conditions. Riemann, in setting up his theory, referred to its future physical meaning, in prophetic words of which one is often reminded in the discussion of the general theory of relativity. In the "question as to the inner ground of the relations of measurement of space," he urges, "the remark can be applied that in a discrete manifold the principle of measurement is already contained in the concept of this manifold, but in the case of a continuous manifold it must come from elsewhere. Either the real lying at the basis of space must be a discrete manifold or the basis of measurement must be sought outside it in binding forces working upon it. The answer to this question can only be found by proceeding from the conception of phenomena, founded by Newton and hitherto verified by experience and gradually reshaping this by facts that cannot be explained

from it; investigations, which, like the one made here, proceed from universal concepts, can only serve to the effect that these works are not hindered by limitations of concepts and the progress in knowledge of the connection of things not hindered by traditional prejudices." (77.) What is here demanded is thus full freedom for the construction of geometrical concepts and hypotheses because only thereby can physical thought attain also full effectiveness, and face all future problems resulting from experience with an assured and systematically perfected instrument. But this connection is expressed, in the case of Riemann, in the language of Herbartian *realism*. At the basis of the pure form of geometrical space a real is to be found in which is to be sought the ultimate cause for the inner relations of measurement of this space. If we carry out, however, with reference to this formulation of the problem, the critical, "Copernican," revolution and thus conceive the question so that a real does not appear as a ground of space but so that space appears as an ideal ground in the construction and progress of knowledge of reality, there results for us at once a characteristic transformation. Instead of regarding "space" as a self-existent real, which must be explained and deduced from "binding forces" like other realities, we ask now rather whether the *a priori* function, the universal ideal relation, that we call "space" involves possible formulations and among them such as are proper to offer an exact and exhaustive account of certain physical relations, of certain "fields of force." The development of the general theory of relativity has answered this question in the affirmative; it has shown what appeared to Riemann as a geometrical hypothesis, as a mere possibility of thought, to be an organ for the knowledge of reality. The Newtonian dynamics is here resolved into pure kinematics and this kinematics ultimately into

geometry. The content of the latter must indeed by broadened and the "simple" Euclidean type of geometrical axioms must be replaced by a more complex type; but in compensation we advance a step further into the realm of being, *i. e.,* into the realm of empirical knowledge, without leaving the sphere of geometrical consideration. By abandoning the form of Euclidean space as an undivided whole and breaking it up analytically and by investigating the place of the particular axioms and their reciprocal dependence or independence, we are led to a system of pure *a priori* manifolds, whose laws thought lays down constructively, and in this construction we possess also the fundamental means for representing the relation of the real structures of the empirical manifold.

The realistic view that the relations of measurement of space must be grounded on certain physical determinations, on "binding forces" of matter, expresses this peculiar double relation one-sidedly and thus, epistemologically regarded, inexactly and unsatisfactorily. For this *metaphysical* use of the category of ground would destroy the *methodological* unity, which should be brought out. What relativistic physics, which has developed strictly and consistently from a theory of space and time measurement, offers us is in fact only the combination, the reciprocal determination, of the metrical and physical elements. In this, however, there is found no one-sided relation of ground and consequent, but rather a purely reciprocal relation, a correlation of the "ideal" and "real" moments, of "matter" and "form," of the geometrical and the physical. In so far as we assume any division at all in this reciprocal relation and take one element as "prior" and fundamental, the other as "later" and derivative, this distinction can be meant only in a logical, not in a real sense. In this sense, we must conceive the pure space-time manifold as the logi-

cal *prius;* not as if it existed and were given in some sense
outside of and before the empirical and physical, but be-
cause it constitutes a principle and a fundamental condi-
tion of all knowledge of empirical and physical relations.
The physicist as such need not reflect on this state of
affairs; for in all the concrete measurements, which he
makes, the spatio-temporal and the empirical manifold is
given always only in the unitary operation of measurement
itself, not in the abstract isolation of its particular concep-
tual elements and conditions.

From these considerations the relation between Eucli-
dean and non-Euclidean geometry appears in a new light.
The real superiority of Euclidean geometry seems at first
glance to consist in its concrete and intuitive determinate-
ness in the face of which all "pseudo-geometries" fade into
logical "possibilities." These possibilities exist only for
thought, not for "being"; they seem analytic plays with
concepts, which can be left unconsidered when we are con-
cerned with experience and with "nature," with the syn-
thetic unity of objective knowledge. When we look back
over our earlier considerations, this view must undergo a
peculiar and paradoxical reversal. Pure Euclidean space
stands, as is now seen, not closer to the demands of em-
pirical and physical knowledge than the non-Euclidean
manifolds but rather more removed. For precisely because
it represents the logically simplest form of spatial construc-
tion it is not wholly adequate to the complexity of content
and the material determinateness of the empirical. Its fun-
damental property of homogeneity, its axiom of the equiva-
lence in principle of all points, now marks it as an abstract
space; for, in the concrete and empirical manifold, there
never is such uniformity, but rather thorough-going dif-
ferentiation reigns in it. If we would create a conceptual
expression for this fact of differentiation in the sphere of

geometrical relations themselves, then nothing remains but
to develop further the geometrical conceptual language
with reference to the problem of the "heterogeneous." We
find this development in the construction of metageometry.
When the concept of the special three-dimensional manifold
with a curvature O is broadened here to the thought of a
system of manifolds with different constant or variable cur-
vatures, a new ideal means is discovered for the mastery of
complex manifolds; new conceptual symbols are created,
not as expressions of things, but of possible relations ac-
cording to law. Whether these relations are realized with-
in phenomena at any place only experience can decide. But
it is not experience that grounds the content of the geo-
metrical concepts; rather these concepts foreshadow it as
methodological anticipations, just as the form of the ellipse
was anticipated as a conic section long before it attained
concrete application and significance in the courses of the
planets. When they first appeared, the systems of non-
Euclidean geometry seemed lacking in all empirical mean-
ing, but there was expressed in them the intellectual prep-
aration for problems and tasks, to which experience was
to lead later. Since the "absolute differential calcu-
lus," which was grounded on purely mathematical con-
siderations by Gauss, Riemann and Christoffel, gains a
surprising application in Einstein's theory of gravitation,
the possibility of such an application must be held open
for all, even the most remote, constructions of pure mathe-
matics and especially of non-Euclidean geometry. For it
has always been shown in the history of mathematics that
its complete freedom contains the guarantee and condition
of its fruitfulness. Thought does not advance in the field
of the concrete by dealing with the particular phenomena
like pictures to be united into a single mosaic, but by sharp-
ening and refining its own means of determination while

guided by reference to the empirical and by the postulate of its determinateness according to law. If a proof were needed for this logical state of affairs, the development of the theory of relativity would furnish it. It has been said of the special theory of relativity that it "substituted mathematical constructions for the apparently most tangible reality and resolved the latter into the former." (38, p. 13.) The advance to the general theory of relativity has brought this constructive feature of it more distinctly to light; but, at the same time, it has shown how precisely this resolution of the "tangible" realities has verified and established the connection of theory and experience in an entirely new way. The further physical thought advances and the higher universality of conception it reaches the more does it seem to lose sight of the immediate data, to which the naïve view of the world clings, so that finally there seems no return to these data. And yet the physicist abandons himself to these last and highest abstractions in the certainty and confidence of finding in them reality, *his* reality in a new and richer sense. In the progress of knowledge the deep words of Heraclitus hold that the way upward and the way downward are one and the same: ὁδὸς ἄνω κάτω μίη. Here, too, ascent and descent necessarily belong together: the direction of thought to the universal principles and grounds of knowledge finally proves not only compatible with the direction to the particularity of phenomena and facts, but its correlate and condition.

VII. THE THEORY OF RELATIVITY AND THE PROBLEM
OF REALITY

W E HAVE attempted to show how the new concept
of nature and of the object, which the theory of rela-
tivity establishes, is grounded in the form of physical
thought and only brings this form to a final conclusion
and clarity. Physical thought strives to determine and to
express in pure objectivity merely the natural object, but
it thereby necessarily expresses itself, its own law and its
own principle. Here is revealed again that 'anthropo-
morphism" of all our concepts of nature to which Goethe's
wisdom of old age loved to point. "All philosphy of nature
is still only anthropomorphism, i. e., man, at unity with
himself, imparts to everything that he is not, this unity,
draws it into his unity, makes it one with him himself. . . .
We can observe, measure, calculate, weigh, etc., nature as
much as we will, it is still only our measure and weight, as
man is the measure of all things." Only, after all our pre-
ceding considerations, this "anthropomorphism" itself is
not to be understood in a limited psychological way but in
a universal, critical and transcendental sense. Planck points
out, as the characteristic of the evolution of the system of
theoretical physics, a progressive emancipation from an-
thropomorphic elements, which has as its goal the greatest
possible separation of the system of physics from the indi-
vidual personality of the physicist. (68, p. 7.) But into
this "objective" system, free from all the accidents of in-
dividual standpoint and individual personality, there enter

those universal conditions of system, on which depends the peculiarity of the physical way of formulating problems. The sensuous immediacy and particularity of the particular perceptual qualities are excluded, but this exclusion is possible only through the concepts of space and time, number and magnitude. In them physics determines the most general content of reality, since they specify the direction of physical thought as such, as it were the form of the original physical apperception. In the formulation of the theory of relativity this reciprocal relation has been confirmed throughout. The principle of relativity has at once an objective and a subjective, or methodological meaning. The "postulate of the absolute world," which it involves according to an expression of Minkowski, is ultimately a postulate of absolute method. The general relativity of all places, times and measuring rods must be the last word of physics, because "relativization," the resolution of the natural object into pure relations of measurement constitutes the kernel of physical *procedure,* the fundamental cognitive function of physics.

If we understand, however, how, in this sense, the affirmation of relativity develops with inner consequence and necessity out of the very form of physics, a certain critical limitation of this affirmation also appears. The postulate of relativity may be the purest, most universal and sharpest expression of the physical concept of objectivity, but this concept of the *physical* object does not coincide, from the standpoint of the general criticism of knowledge, with reality absolutely. The progress of epistemological analysis is shown in that the assumption of the simplicity and oneness of the concepts of reality is recognized more and more as an illusion. Each of the original directions of knowledge, each interpretation, which it makes of phenomena to combine them into the unity of a theoretical

connection or into a definite unity of meaning, involves a
special understanding and formulation of the concept of
reality. There result here not only the characteristic dif-
ferences of meaning in the objects of science, the distinc-
tion of the "mathematical" object from the "physical" ob-
ject, the "physical" from the "chemical," the "chemical"
from the biological," but there occur also, over against the
whole of *theoretical* scientific knowledge, other forms and
meanings of independent type and laws, such as the ethical,
the aesthetic "form." It appears as the task of a truly uni-
versal criticism of knowledge not to level this manifold,
this wealth and variety of forms of knowledge and under-
standing of the world and compress them into a purely
abstract unity, but to leave them standing as such. Only
when we resist the temptation to compress the totality of
forms, which here result, into an *ultimate* metaphysical
unity, into the unity and simplicity of an absolute "world
ground" and to deduce it from the latter, do we grasp its
true concrete import and fullness. No individual form can
indeed claim to grasp absolute "reality" as such and to
give it complete and adequate expression. Rather if the
thought of such an ultimate definite reality is conceivable at
all, it is so only as an Idea, as the problem of a totality
of determination in which each particular function of
knowledge and consciousness must coöperate according to
its character and within its definite limits. If one holds
fast to this general view, there results even within the pure
concepts of nature a possible diversity of approaches of
which each one can lay claim to a certain right and char-
acteristic validity. The "nature" of Goethe is not the same
as that of Newton, because there prevail, in the original
shaping of the two, different principles of form, types of
synthesis, of the spiritual and intellectual combination of
the phenomena. Where there exist such diversities in fun-

damental *direction* of consideration, the *results* of consideration cannot be directly compared and measured with each other. The naïve realism of the ordinary view of the world, like the realism of dogmatic metaphysics, falls into this error, ever again. It separates out of the totality of possible concepts of reality a single one and sets it up as a norm and pattern for all the others. Thus certain necessary formal points of view, from which we seek to judge and understand the world of phenomena, are made into things, into absolute beings. Whether we characterize this ultimate being as "matter" or "life," "nature" or "history," there always results for us in the end confusion in our view of the world, because certain spiritual functions, that coöperate in its construction, are excluded and others are over-emphasized.

It is the task of systematic philosophy, which extends far beyond the theory of knowledge, to free the idea of the world from this one-sidedness. It has to grasp the *whole system* of symbolic forms, the application of which produces for us the concept of an ordered reality, and by virtue of which subject and object, ego and world are separated and opposed to each other in definite form, and it must refer each individual in this totality to its fixed place. If we assume this problem solved, then the rights would be assured, and the limits fixed, of each of the particular forms of the concept and of knowledge as well as of the general forms of the theoretical, ethical, aesthetic and religious understanding of the world. Each particular form would be "relativized" with regard to the others, but since this "relativization" is throughout reciprocal and since no single form but only the systematic totality can serve as the expression of "truth" and "reality," the limit that results appears as a thoroughly immanent limit, as one that is

removed as soon as we again relate the individual to the
system of the whole.

We trace the general problem, which opens up here,
no further but use it merely to designate the lim-
its, that belong to any, even the most universal, *physical*
formulation of problems, because these limits are neces-
sarily grounded in the concept and essence of this way of
formulating the question. All physics considers phenom-
ena under the standpoint and presupposition of their meas-
urability. It seeks to resolve the structure of being and
process ultimately into a pure structure or order of num-
bers. The theory of relativity has brought this fundamen-
tal tendency of physical thought to its sharpest expression.
According to it the procedure of every physical "explana-
tion" of natural process consists in coördinating, to each
point of the space-time continuum, four numbers, x_1, x_2, x_3,
x_4, which possess absolutely no direct physical meaning but
only serve to enumerate the points of the continuum "in
a definite, but arbitrary way." (18, p. 64.) The ideal,
with which scientific physics began with Pythagoras and
the Pythagoreans, finds here its conclusion; all qualities,
including those of pure space and time, are translated into
pure numerical values. The logical postulate contained
in the concept of number, which gives this concept its char-
acteristic form, seems now fulfilled in a degree not to be
surpassed; all sensuous and intuitive heterogeneity has
passed into pure homogeneity. The classical mechanics
and physics seeks to reach this immanent goal of concep-
tual construction by relating the manifold of the sensuously
given to the homogeneous and absolutely uniform time. All
difference of sensation is hereby reduced to a differ-
ence of motions; all possible variety of content is re-
solved into a mere variety of spatial and temporal positions.
But the ideal of strict homogeneity is not reached here

since there are still always two fundamental forms of the homogeneous itself that are opposed to each other as pure space and pure time. The theory of relativity in its development advances beyond this opposition also; it seeks to resolve not only the differences of sensation but also those between spatial and temporal determinations into the unity of numerical determinations. The particularity of each "event" is expressed by the four numbers x_1, x_2, x_3, x_4, whereby these numbers among themselves have reference to no inner differences, so that some of them x_1, x_2, x_3, cannot be brought into a special group of "spatial" coördinates and contrasted with the time coördinate" x_4. Thus all differences belonging to spatial and temporal apprehension in subjective consciousness seem to be consistently set aside in the same way that nothing of the subjective visual sensation enters into the physical concept of light and color.[31] Not only are all spatial and temporal values exchangeable with each other, but all inner differences of the temporal itself, unavoidable for the subjective consciousness, all differences of *direction,* which we designate by the words "past" and "future," are cancelled. The direction into the past and that into the future are distinguished from each other in this form of the concept of the world by nothing more than are the $+$ and $-$ directions in space, which we can determine by arbitrary definition. There remains only the "absolute world" of Minkowski; the world of physics changes from a *process* in three-dimensional world in which time is replaced as a variable magnitude by the imaginary "ray of light" (*Lichtweg*) $(x_4 = \sqrt{-1}\, c\, t)$.[32]

This transformation of the time-value into an imaginary numerical value seems to annihilate all "reality" and qualitative determinateness, which time possesses as the

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[31] On this latter point *cf.* now Planck, *Das Wesen des Lichts* (71).
[32] *Cf.* Minkowski (54, p. 62ff.); Einstein (18, p. 82f.).

"form of the inner sense," as the form of immediate experience. The "stream of process," which, psychologically, constitutes consciousness and distinguishes it as such, stands still; it has passed into the absolute rigidity of a mathematical cosmic formula. There remains in this formula nothing of that form of time, which belongs to all our experience as such and enters as an inseparable and necessary factor into all its content.[33] But, paradoxical as this result seems from the standpoint of this experience, it expresses only the course of mathematical and physical objectification, for, to estimate it correctly from the epistemological standpoint, we must understand it not in its mere result, but as a process, a method. In the resolution of subjectively experienced qualites into pure objective numerical determinations, mathematical physics is bound to no fixed limit. It must go its way to the end; it can stop before no form of consciousness no matter how original and fundamental; for it is precisely its specific cognitive task to translate everything enumerable into pure number, all quality into quantity, all particular forms into a universal order and it only "conceives" them scientifically by virtue of this transformation. Philosophy would seek in vain to bid this tendency halt at any point and to declare *ne plus ultra*. The task of philosophy must rather be limited to conceiving this *meaning* in its logical dependency by recognizing fully the logical *meaning* of the mathematical and physical concept of objectivity. All particular physical theories including the theory of relativity receive their definite meaning and import only through the unitary cognitive will of physics, which stands back of them. The moment that we transcend the field of physics and change not the means but the very goal of knowledge, all particular concepts assume a new aspect and form. Each of these concepts means something different, depending on the general "modality"

[33] *Cf., e. g.,* J. Cohn (14, p. 228ff.).

of consciousness and knowledge with which it stands and from which it is considered.   Myth and scientific knowledge, the logical and the aesthetic consciousness, are examples of such diverse modalities.   Occasionally concepts of the same name, but by no means of the same meaning, meet us in these different fields.   The conceptual relation, which we generally call "cause" and "effect" is not lacking to mythical thought, but here its meaning is specifically distinct from the meaning that it receives in scientific, and in particular, in mathematical and physical thought.   In a similar way, all the fundamental concepts undergo a characteristic intellectual change of meaning when we trace them through the different fields of intellectual consideration. Where the copy theory of knowledge seeks a simple identity, the functional theory of knowledge sees complete diversity, but, indeed, at the same time complete correlation of the individual forms.[34]

If we apply these considerations to the concepts of space and time, then it is obvious what the transformation of these concepts in modern physics means, in its philosophical import, and what it cannot mean.   The content of physical deductions cannot, without falling into the logical error of a μετάβασις εἰς ἄλλο γένος be simply carried over into the language of fields whose structure rests on a totally different structural principle.   Thus, what space and time are as immediate contents of experience and as they offer themselves to our psychological and phenomenological analysis ìs unaffected by the use we make of them in the determination of the object, in the course of objective conceptual knowledge.   The distance between these two types of consideration and conception is only augmented by the theory of relativity and thus only made known more distinctly, but is not first produced by it.   Rather it is clear

[34] I am aware of the fragmentary character of these suggestions; for their supplementation and more exact proof I must refer to some subsequent more exhaustive treatment.   Cf. also the essay *Goethe und die mathematische Physik* (11).

that even to attain the first elements of mathematical and
physical knowledge and of the mathematical and physi-
cal object we assume that characteristic transforma-
tion of "subjective" phenomenal space and of "subjective"
phenomenal time, which leads, in its ultimate consequences,
to the results of the general theory of relativity. From the
standpoint of strict sensualism too, it is customary to admit
this transformation, this opposition between the "physio-
logical" space of our sensation and presentation and the
purely "metrical" space, which we make the basis of geom-
etry. The latter rests on the assumption of the equiva-
lence of all places and directions, while for the former the
distinction of places and directions and the marking out of
one above the others is essential. The space of touch, like
that of vision, is anistropic and inhomogeneous, while
metrical Euclidean space is distinguished by the postulate
of isotropism and homogeneity. Compared with "metri-
cal" time, physiological time shows the same characteristic
variations and differences of meaning; one must, as Mach
himself urges, as clearly distinguish between the imme-
diate sensation of duration and the measuring number as
between the sensation of warmth and temperature.[35]

[35] Mach (50, p. 331ff., 415ff.). If, with Schlick (79, p. 51ff.), one would
call the psychological space of sensation and presentation the space of intui-
tion, and contrast with it physical space as a conceptual construction, no objec-
tion could be made against this as a purely terminological determination; but
one must guard against confusing this use of the word "intuition" with the
Kantian, which rests on entirely different presuppositions. When Schlick
sees in the insight that objective physical time has just as little to do with the
intuitive experience of duration as the three-dimensional order of objective
space with optical or "haptical" extension, "the kernel of truth in the Kantian
doctrine of the subjectivity of time and space," and when he, on the other
hand combats, on the basis of this distinction, the Kantian concept of "pure
intuition," this rests on a psychological misunderstanding of the meaning of the
Kantian concepts. The space and time of pure intuition are for Kant never
sensed or perceived space or time, but the "mathematical" space and time of
Newton; they are themselves constructively generated, just as they form the
presupposition and foundation of all further mathematical and physical con-
struction. In Kant's thought, "pure intuition" plays the rôle of a definite fun-
damental *method of objectification;* it coincides in no way with "subjective,"
i. e., psychologically experienceable time and space. When Kant speaks of the
subjectivity of space and time, we must never understand experiential sub-
jectivity but their "transcendental" subjectivity as conditions of the possibility
of "objective," *i. e.,* of objectifying empirical knowledge. (*Cf.* also the signifi-
cant remarks of Selliens against Schlick; 81, p. 19, 39.)

This contrast between subjective, "phenomenal" space and time, on the one hand, and objective and mathematical space and time, on the other, comes to light with special distinctness, when one considers a property which seems at first glance to be common to them.   Of both we are accustomed to predicate the property of *continuity,* but we understand thereby, more closely considered, in the two cases something wholly different.   The continuity, which we ascribe to time and processes in it on the basis of the form of our experience, and that which we define in mathematical concepts by certain constructive methods of analysis, not only do not coincide but they differ in their essential moments and conditions.   The experiential continuity affirms that each temporal content is given to us only in the way of certain characteristic *"wholes,"* which can not be resolved into ultimate simple *"elements"*; analytic continuity demands reduction to such elements.   The first takes time and duration as "organic" unities in which according to the Aristotelian definition, "the whole precedes the parts"; the second sees in them only an infinite *totality* of parts, of particular sharply differentiated *temporal points.* In the one case, the continuity of becoming signifies that living flux, that is given to our consciousness only as a flux, as a transition, but not as separated and broken up into discrete parts; in the other, it is demanded that we continue our analysis beyond all limits of empirical apprehension; it is demanded that we do not allow the division of elements to cease where sensuous perception, which is bound to definite but accidental limits in its capacity for discrimination, allows it to end, but that we follow it purely intellectually *ad infinitum.*   What the mathematician calls the "continuum" is thus never the purely experiential quality of "continuity," of which there is no longer possible any further "objective" definition, but it is a purely conceptual construction, which he puts in the place of the latter.   Here

too he must follow his universal method; he must reduce
the quality of continuity to mere number, *i. e.,* precisely to
the fundamental form of all intellectual *discreteness.* (Cf.
6, p. 21.) The only continuum he knows and the one to
which he reduces all others, is always the continuum of
real numbers which modern analysis and theory of groups
seek, as is known, to construct strictly conceptually with
renunciation in principle of any appeal to the "intuition"
of space and time. The continuum thus considered, as
Henri Poincaré especially has urged with all emphasis is
nothing but a totality of individuals, which are conceived
in a definite order and are given indeed in infinite number,
of which each one is opposed to the others as something
separate and external. We are here no longer concerned
with the ordinary view, according to which there exists
between the elements a sort of "inner bond" by which they
are connected into a whole, so that, *e. g.,* the point does not
precede the line, but the line the point. "Of the famous
formula, that the continuum is the unity of the manifold,"
concludes Poincaré, "there remains only the manifold,—
the unity has disappeared. The analysts are nevertheless
right when they define continuity as they do, for in all their
inferences they are concerned, in so far as they claim rigor,
only with this concept of the continuous. But this circum-
stance suffices to make us attentive to the fact that the true
mathematical continuum is something totally different from
that of the physicist and the metaphysician." (72, p. 30.)
In so far as physics is an objectifying science working with
the conceptual instruments of mathematics, the physical
continuum is conceived by it as related to and exactly cor-
related with the mathematical continuum of pure num-
bers. But the "metaphysical" continuum of the pure and
original "subjective" form of experience can never be com-
prehended in this way, for the very *direction* of mathemati-
cal consideration is such that, instead of leading to this

form, it continually leads away from it.   The critical theory of knowledge, which does not have to select from among the different sorts of knowledge, but merely to establish what each of them "is" and means, can make no normative decision as to the opposite aspects under which the continuum here appears, but its tasks consists in defining the two with reference to each other in utmost distinctness and clarity.   Only by such a delimitation can be reached, on the one hand, the goal of phenomenological analysis of the temporal and spatial consciousness, and on the other hand, the goal of the exact foundation of mathematical analysis and its concepts of space and time.   "With regard to the objection," a modern mathematical author concludes his investigation of the continuum, "that nothing is contained in the intuition of the continuum of the logical principles that we must adduce in the exact definition of the concept of the real number, we have taken account of the fact that what can be found in the intuitive continuum and in the mathematical world of concepts are so alien to each other, that the demand that the two coincide must be rejected as absurd.   In spite of this, those abstract schemata, which mathematics offers us, are helpful in rendering possible an exact science of fields of objects in which continua play a rôle.   The exact temporal or spatial point does not lie in the given (phenomenal) duration or extension as an ultimate indivisible element, but only reason reaching through this can grasp these ideas and they crystallize into full determinateness only in connection with the purely formal arithmetical and analytical concept of the real number." [36]

If we bear in mind this state of affairs, the deductions of the theory of relativity in its determination of the four dimensional space and time continuum lose the appearance of paradox, for it is seen that they are only the final conse-

[36] Weyl, 84, p. 83, 71.

quence and working out of the fundamental methodic idea on which rests mathematical analysis in general. But the question as to which of the two forms of space and time, the psychological or the physical, the space and time of immediate experience or of mediate conception and knowledge, expresses the *true* reality has lost fundamentally for us all definite meaning. In the complex that we call our "world," that we call the being of our ego and of things, the two enter as equally unavoidable and necessary moments. We can cancel neither of them in favor of the other and exclude it from this complex, but we can refer each to its definite *place* in the whole. If the physicist, whose problem consists in objectification, affirms the superiority of "objective" space and time over "subjective" space and time; if the psychologist and the metaphysician, who are directed upon the totality and immediacy of experience draw the opposite conclusion; then the two judgments express only a false "absolutization" of the norm of knowledge by which each of them determines and measures "reality." In which direction this "absolutization" takes place and whether it is directed on the "outer" or the "inner" is a matter of indifference to the purely epistemological judgment. For Newton it was certain that the absolute and mathematical time, which by its nature flowed uniformly, was the "true" time of which all empirically given temporal determination can offer us only a more or less imperfect copy; for Bergson, this "true" time of Newton is a conceptual fiction and abstraction, a barrier, which intervenes between our apprehension and the original meaning and import of reality. But it is forgotten that what is here called absolute reality, *dureé rèelle,* is itself no absolute but only signifies a standpoint of consciousness opposed to that of mathematics and physics. In the one case, we seek to gain a unitary and exact measure for all objective process, in the other we are concerned in retain-

ing this process itself in its pure qualitative character, in its concrete fulness and subjective inwardness and "contentuality." The two standpoints can be understood in their meaning and necessity; neither suffices to include the actual whole of being in the idealistic sense of "being for us." The symbols that the mathematician and physicist take as a basis in their view of the external and the psychologist in his view of the inner, must both be understood as *symbols*. Until this has come about, the true philosophical view, the view of the *whole,* is not reached, but a partial experience is hypostasized into the whole. From the standpoint of mathematical physics, the total content of the immediate qualities, not only the differences of sensation, but those of spatial and temporal consciousness, is threatened with complete annihilation; for the metaphysical psychologist, conversely, all reality is reduced to this immediacy, while every mediate conceptual cognition is given only the value of an arbitrary convention produced for the purposes of our action. But both views prove, in their absoluteness, rather perversions of the full import of being, *i. e.,* of the full import of the *forms* of knowledge of the self and the world. While the mathematician and the mathematical physicist stand in danger of permitting the real world to be identified with the world of their *measures,* the metaphysical view, in seeking to narrow mathematics to practical goals, loses the sense of its purest and deepest *ideal* import. It violently closes the door against what, according to Plato, constitutes the real meaning and the real value of mathematics; that, namely, "by each of these cognitions an *organ of the soul* is purified and strengthened, which under other occupations is lost and blinded; for its preservation is more important than that of a thousand eyes: for by this alone is the truth seen." And been the two poles of consideration, which we find here, there stand the manifold concepts of truth of the different concrete sciences—

and therewith their concepts of space and time. *History,* to set up its temporal measure, cannot do without the methods of the objectifying sciences: chronology is founded on astronomy and through this on mathematics. But the time of the historian is nevertheless not identical with that of the mathematician and physicist, but possesses in contrast to it a peculiar concrete form. In the concept of time of history, the "objective" content of knowledge and the "subjective" experiential content enter into a new characteristic reciprocal relation. An analogous relation is presented, when we survey the aesthetic meaning and shaping of the forms of space and time. Painting presupposes the objective laws of perspective, architecture the laws of statics, but the two serve here only as material out of which develops the unity of the picture and of the architectural spatial form, on the basis of the original artistic laws of form. For music, too, the Pythagoreans sought a connection with pure mathematics, with pure number; but the unity and rythmical division of a melody rests on wholly different structural principles than those on which we construct time in the sense of the unity of objective physical processes of nature. What space and time truly *are* in the philosophical sense would be determined if we succeeded in surveying completely this wealth of nuances of intellectual meaning and in assuring ourselves of the underlying formal law under which they stand and which they obey. The theory of relativity cannot claim to bring this philosophical problem to its solution; for, by its evolution and scientific tendency from the beginning, it is limited to a definite particular motive of the concepts of space and time. As a physical theory it merely develops the meaning that space and time possess in the system of our empirical and physical measurements. In this sense, final judgment on it belongs exclusively to physics. In the course of its history, physics will have to decide whether the world- picture of

the theory of relativity is securely founded theoretically and whether it finds complete experimental verification. Its decision on this, epistemology cannot anticipate; but even now it can thankfully receive the new incitements which this theory has given the general doctrine of the principles of physics.

AUTHOR'S NOTE

The above essay, of which this is the concluding section, does not claim to give a complete account of the philosophical problems raised by the theory of relativity. I am aware that the new problems presented to the general criticism of knowledge by this theory can only be mastered by the gradual and common work of physicists and philosophers; here I was merely concerned with beginning this work, with stimulating discussion, and, where possible, guiding it into definite methodic paths, in contrast to the uncertainty of judgment which still reigns. The purpose of this writing would be attained if it succeeded in preparing for a mutual understanding between the philosopher and the physicist on questions, concerning which they are still widely separated. That I was concerned, in purely epistemological matters, also, to hold myself in closest contact with scientific physics and that the writings of the leading physicists of the past and present have everywhere essentially helped to determine the intellectual orientation of the preceding investigation, will be gathered from the exposition. The bibliography, which follows, however, makes no claim to actual completeness; in it only such works are adduced as have been repeatedly referred to and intensively considered in the course of the exposition.

Albert Einstein read the above essay in manuscript and improved it by his critical comments; I cannot let it go out without expressing here also my hearty thanks to him.

ERNST CASSIRER.

UNIVERSITY OF HAMBURG.

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